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CAPUTO-TYPE MODIFICATION OF THE ERDÉLYI-KOBER FRACTIONAL DERIVATIVE

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Dedicated to Professor Michele Caputo
on the occasion of his 80th anniversary

Abstract

The Caputo fractional derivative is one of the most used definitions of a fractional derivative along with the Riemann-Liouville and the Grünwald-Letnikov ones. Whereas the Riemann-Liouville definition of a fractional derivative is usually employed in mathematical texts and not so frequently in applications, and the Grünwald-Letnikov definition – for numerical approximation of both Caputo and Riemann-Liouville fractional derivatives, the Caputo approach appears often while modeling applied problems by means of fractional derivatives and fractional order differential equations.

In the mathematical texts and applications, the so called Erdélyi-Kober (E-K) fractional derivative, as a generalization of the Riemann-Liouville fractional derivative, is often used, too. In this paper, we investigate some properties of the Caputo-type modification of the Erdélyi-Kober fractional derivative. The relation between the Caputo-type modification of the E-K fractional derivative and the classical E-K fractional derivative is the same as the relation between the Caputo fractional derivative and the Riemann-Liouville fractional derivative, i.e. the operations of integration and differentiation are interchanged in the corresponding definitions. Here, some new properties of the classical Erdélyi-Kober fractional derivative and the respective ones of its Caputo-type modification are presented together.

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1. Introduction

The Caputo-type modification of the Erdélyi-Kober fractional derivative, we deal with in this paper, was introduced for the first time by Gorenflo, Luchko and Mainardi in [7] in connection with their investigation of the scale-invariant solutions of the diffusion-wave equation. The time-fractional diffusion-wave equation is obtained from the classical diffusion or wave equation by replacing the first- or second-order time derivative by a fractional derivative of order α $(0 < \alpha \leq 2)$. Partial differential equations of fractional order have been successfully used for modelling relevant physical processes (see, for example, Caputo [2], Chechkin et al. [3], Freed et al. [5], Giona and Roman [6], Hilfer [8], [9], [10], Kilbas et al. [12], Mainardi [16], Mainardi and Tomirotti [17], Mainardi et al. [18], [19], Matignon and Montseny [20], Metzler et al. [21], Nigmatullin [22], Pipkin [23], Podlubny [24] and references there). In the applications, special types of solutions, which are invariant under some subgroup of the full symmetry group of the given equation (or for a system of equations) are especially important.

In [7], the authors considered the diffusion-wave equation with the Caputo fractional derivative of the order α, $n-1 < \alpha \leq n$, $n \in I\!\!N$. For the scale-invariant solutions of this equation, they deduced a fractional differential equation in the following form:

$$({}_*P_{2/\alpha}^{\gamma-n+1,\alpha} v)(y) = \mathcal{D} v''(y), \tag{1}$$

where the operator in the left-hand side is the Caputo-type modification of the Erdélyi-Kober fractional differential operator that will be defined in the last section of the paper, $y = xt^{-\alpha/2}$ is the similarity variable (t and x being the time- and temporal variables of the diffusion-wave equation) and γ is a similarity parameter. The exact solution of the equation (1) was given in [7] in terms of the Wright function

$$W_{\rho,\mu}(z) := \sum_{k=0}^{\infty} \frac{z^k}{k!\Gamma(\rho k + \mu)}, \qquad \rho \in I\!\!R,\ \mu \in \mathbb{C}, \tag{2}$$

and the generalized Wright function

$$W_{(\mu,a),(\nu,b)}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(a+\mu k)\Gamma(b+\nu k)}, \qquad \mu,\nu \in I\!\!R,\ a,b \in \mathbb{C}. \tag{3}$$

Wright himself has investigated the function (3) in the case $\mu > 0$, $\nu > 0$ in [27]. If $a = \mu = 1$ or $b = \nu = 1$, respectively, then it is reduced to the

Wright function (2). For the properties of the generalized Wright function for other values of the parameters see [7] and [15].

Since the aim of the paper [7] was mainly to investigate the scale-invariant solutions of the diffusion-wave equation, the introduced Caputo-type modification of the Erdélyi-Kober fractional differential operator was not considered there in details. In this paper, we present and prove some basic properties of this operator and suggest some open problems for further research.

The plan of the rest of the paper is as follows. In the second section we introduce the Erdélyi-Kober fractional integrals and investigate their action on the space of functions C_α that consists of all functions that are continuous on the half-axis $(0,\infty)$ but can have a power singularity at the origin. The third section deals with the properties of the Erdélyi-Kober fractional derivatives. In particular, the composition of the E-K fractional derivative and the E-K fractional integral and the composition of the E-K fractional integral and the E-K fractional derivative are evaluated simultaneously, in suitable functional spaces. In Section 4 we define the Caputo-type modification of the Erdélyi-Kober fractional derivative and investigate some of its properties. In particular, the composition of the E-K fractional integral and the Caputo-type modification of the E-K fractional derivative is calculated, and the conditions under which the Caputo-type modification of the E-K fractional derivative coincides with the classical E-K fractional derivative are given. In conclusion, we present some open problems related with the Caputo-type modification of the Erdélyi-Kober fractional derivative that will be considered elsewhere.

2. Erdélyi-Kober fractional integrals

The right- and the left-hand sided Erdélyi-Kober (E-K) fractional integrals of the orders δ and α, respectively, are defined by

$$(I_\beta^{\gamma,\delta} f)(x) = \frac{\beta}{\Gamma(\delta)} x^{-\beta(\gamma+\delta)} \int_0^x (x^\beta - t^\beta)^{\delta-1} t^{\beta(\gamma+1)-1} f(t)dt,\ \ \delta,\beta>0, \gamma\in I\!R, \tag{4}$$

$$(J_\beta^{\tau,\alpha} f)(x) = \frac{\beta}{\Gamma(\alpha)} x^{\beta\tau} \int_x^\infty (t^\beta - x^\beta)^{\alpha-1} t^{-\beta(\tau+\alpha-1)-1} f(t)dt,\ \ \alpha,\beta>0, \tau\in I\!R. \tag{5}$$

These operators have been used by many authors, in particular, to obtain solutions of the single, dual and triple integral equations possessing special functions of mathematical physics as their kernels. For the theory and applications of the Erdélyi-Kober fractional integrals see e.g. [26], and more recently – [11], [12], [13], [25], and [28]. In particular, in [11], [13] the following important properties of the operator (4) have been proved (of course, the left-hand sided Erdélyi-Kober integral (5) possesses similar properties, but in this paper we mainly restrict our attention to the right-hand sided operator (4)):

$$(I_\beta^{\gamma,\delta} x^{\lambda\beta} f)(x) = x^{\lambda\beta}(I_\beta^{\gamma+\lambda,\delta} f)(x), \tag{6}$$

$$(I_\beta^{\gamma,\delta} I_\beta^{\gamma+\delta,\alpha} f)(x) = (I_\beta^{\gamma,\delta+\alpha} f)(x), \tag{7}$$

$$(I_\beta^{\gamma,\delta} I_\beta^{\alpha,\eta} f)(x) = (I_\beta^{\alpha,\eta} I_\beta^{\gamma,\delta} f)(x). \tag{8}$$

For $\beta = 1$ the operator (4) is reduced to the Kober operator

$$(K^{\gamma,\delta} f)(x) = \frac{x^{-\gamma-\delta}}{\Gamma(\delta)} \int_0^x (x-t)^{\delta-1} t^\gamma f(t) dt, \delta, \beta > 0, \tag{9}$$

that was introduced for the first time by Kober in [14]. In particular, Kober proved ([14]) the following important result:

Theorem 2.1. *Let* $\Re(\gamma - s) > -1,\ f \in L_p(0,\infty),\ 1 \le p \le 2,\ \gamma > -\frac{1}{q},\ \frac{1}{p} + \frac{1}{q} = 1$. *Then the formula*

$$(\mathcal{M}\, K^{\gamma,\delta} f)(s) = \frac{\Gamma(1+\gamma-s)}{\Gamma(1+\gamma+\delta-s)} (\mathcal{M}\, f)(s) \tag{10}$$

holds true, where

$$(\mathcal{M} f)(s) := \int_0^{+\infty} f(t) t^{s-1} dt \tag{11}$$

denotes the Mellin integral transform of a function f.

Formula (10) can be extended to the case of arbitrary β, i.e. for the Erdélyi-Kober operator (4) (see e.g. [12], [28]):

$$(\mathcal{M}\, I_\beta^{\gamma,\delta} f)(s) = \frac{\Gamma(1+\gamma-s/\beta)}{\Gamma(1+\gamma+\delta-s/\beta)} (\mathcal{M}\, f)(s). \tag{12}$$

For $\gamma = 0$, the Kober operator (9) is reduced to the Riemann-Liouville fractional integral with a power weight:

$$(K^{0,\delta} f)(x) = \frac{x^{-\delta}}{\Gamma(\delta)} \int_0^x (x-t)^{\delta-1} f(t) dt, \delta > 0. \tag{13}$$

In this paper, we consider the Erdélyi-Kober fractional integral (4) in a special space of functions that was introduced for the first time by Dimovski in [4] (see also [13] and [28]):

DEFINITION 2.1. The space of functions $C_\alpha,\ \alpha \in I\!R$ consists of all functions $f(x),\ x > 0$, that can be represented in the form $f(x) = x^p f_1(x)$ with $p > \alpha$ and $f_1 \in C([0,\infty))$.

It is obvious that the space C_α is linear and the following inclusion for the set of spaces $C_\alpha,\ \alpha \in I\!R$ holds true:

$$C_\alpha \subseteq C_\beta,\ \alpha \geq \beta. \tag{14}$$

THEOREM 2.2. *Let $\alpha \geq -\beta(\gamma+1)$. Then the Erdelyi-Kober fractional integration operator (4) is a linear map of the space C_α into itself, i.e.*

$$I_\beta^{\gamma,\delta} : C_\alpha \to C_\alpha. \tag{15}$$

To prove the theorem, we first represent the Erdelyi-Kober fractional integration operator (4) in the following form by using the change of the variables $t = x\tau^{1/\beta}$:

$$(I_\beta^{\gamma,\delta} f)(x) = \int_0^1 \frac{(1-\tau)^{\delta-1}\tau^\gamma}{\Gamma(\delta)} f(x\tau^{1/\beta})d\tau. \tag{16}$$

For a function f from the space C_α, this representation leads to

$$(I_\beta^{\gamma,\delta} f)(x) = \int_0^1 \frac{(1-\tau)^{\delta-1}\tau^\gamma}{\Gamma(\delta)} f(x\tau^{1/\beta})d\tau \tag{17}$$

$$= x^p \int_0^1 \frac{(1-\tau)^{\delta-1}\tau^{\gamma+p/\beta}}{\Gamma(\delta)} f_1(x\tau^{1/\beta})d\tau = x^p f_2(x),$$

where the function f_1 is continuous on the interval $[0,\infty)$. We have then the following estimate for $0 \leq x \leq X,\ 0 < \tau < 1$:

$$\left| \frac{(1-\tau)^{\delta-1}\tau^{\gamma+p/\beta}}{\Gamma(\delta)} f_1(x\tau^{1/\beta}) \right| \leq A \frac{(1-\tau)^{\delta-1}\tau^{\gamma+p/\beta}}{\Gamma(\delta)}. \tag{18}$$

For $\alpha \geq -\beta(\gamma+1)$ and $p > \alpha$ we have $\gamma + p/\beta > -1$. It follows then from the estimate (18) and the representation (17) that the function f_2 is determined by an uniformly convergent integral with respect to x in any closed interval $[0, X]$. Consequently, the function f_2 is a continuous one on the interval $[0, X]$ and, since we can choose any $X > 0$, on the interval $[0,\infty)$ as well, that finishes the proof of the theorem. ■

For analogous result, in the case of composition of commuting E-K fractional integrals $I_{\beta_k}^{\gamma_k,\delta_k}$, $k = 1,\dots,m$, one can see [13], Th. 1.2.15.

3. Erdélyi-Kober fractional derivatives

In this section we consider the right- and the left-hand sided Erdélyi-Kober fractional derivatives of orders δ and α, respectively (see e.g. [12], [13], [25], [28]). Let $n-1<\delta\le n,\ n\in I\!N$ and $m-1<\alpha\le m,\ m\in I\!N$. The operator

$$(D_\beta^{\gamma,\delta}f)(x):=\prod_{j=1}^{n}(\gamma+j+\frac{1}{\beta}x\frac{d}{dx})(I_\beta^{\gamma+\delta,n-\delta}f)(x) \tag{19}$$

is called the right-hand sided Erdélyi-Kober (E-K) fractional derivative of order δ. The left-hand sided E-K fractional derivative of order α is defined, resp., by

$$(P_\beta^{\tau,\alpha}f)(x):=\prod_{j=0}^{m-1}(\tau+j-\frac{1}{\beta}x\frac{d}{dx})(J_\beta^{\tau+\alpha,m-\alpha}f)(x). \tag{20}$$

In the formulae (19) and (20) the operators $I_\beta^{\gamma,\delta}$ and $J_\beta^{\tau,\alpha}$ are the right- and the left-hand sided Erdélyi-Kober fractional integrals of orders δ and α, respectively, defined by (4) and (5). In this paper, we deal mainly with the right-hand sided Erdélyi-Kober derivative $D_\beta^{\gamma,\delta}$. The case of the left-hand sided derivative can be considered by analogy.

For the functions from the space $C_\alpha,\ \alpha\ge-\beta(\gamma+1)$, the right-hand sided E-K fractional derivative is a left-inverse operator to the right-hand sided E-K fractional integration operator (4). Let us prove this fact, i.e., that

$$(D_\beta^{\gamma,\delta}\,I_\beta^{\gamma,\delta}f)(x)\equiv f(x),\quad f\in C_\alpha. \tag{21}$$

Using the definition of the E-K fractional derivative and the property (7) of the E-K fractional integral, we obtain the relation

$$(D_\beta^{\gamma,\delta}\,I_\beta^{\gamma,\delta}f)(x)=\prod_{j=1}^{n}(\gamma+j+\frac{1}{\beta}x\frac{d}{dx})(I_\beta^{\gamma+\delta,n-\delta}I_\beta^{\gamma,\delta}f)(x) \tag{22}$$

$$=\prod_{j=1}^{n}(\gamma+j+\frac{1}{\beta}x\frac{d}{dx})(I_\beta^{\gamma,n}f)(x).$$

Since $n\in I\!N$, the relation

$$\prod_{j=1}^{n}(\gamma+j+\frac{1}{\beta}x\frac{d}{dx})(I_\beta^{\gamma,n}f)(x)\equiv f(x) \tag{23}$$

can be proved by the mathematical induction. For $n = 1$, we have

$$(\gamma+1+\frac{1}{\beta}x\frac{d}{dx})(I_\beta^{\gamma,1}f)(x) = (\gamma+1+\frac{1}{\beta}x\frac{d}{dx})\left(\beta x^{-\beta(\gamma+1)}\int_0^x t^{\beta(\gamma+1)-1}f(t)dt\right)$$

$$= (\gamma+1)\beta x^{-\beta(\gamma+1)}\int_0^x t^{\beta(\gamma+1)-1}f(t)dt + x\frac{d}{dx}\left(x^{-\beta(\gamma+1)}\int_0^x t^{\beta(\gamma+1)-1}f(t)dt\right)$$

$$= (\gamma+1)\beta x^{-\beta(\gamma+1)}\int_0^x t^{\beta(\gamma+1)-1}f(t)dt - \beta(\gamma+1)x^{-\beta(\gamma+1)}\int_0^x t^{\beta(\gamma+1)-1}f(t)dt$$

$$+f(x) \equiv f(x).$$

Now we have to prove the relation (23) for $n = k + 1$ under the assumption that it holds valid for $n = k$. To reduce the case $n = k + 1$ to the case $n = k$, we just have to check the following relation:

$$(\gamma + 1 + k + \frac{1}{\beta}x\frac{d}{dx})(I_\beta^{\gamma,k+1}f)(x) \equiv (I_\beta^{\gamma,k}f)(x).$$

This can be done by direct differentiation in the same way as it has been done for the case $n = 1$ and we omit the long but straightforward calculations. ■

It is known, that in the general case the E-K fractional derivative is not a right-inverse operator to the E-K fractional integration operator (4). To formulate the corresponding result, we introduce a subspace of the space C_α, where both Erdélyi-Kober fractional derivative (19) and its Caputo-type modification (that will be introduced in the next section) are well defined.

Definition 3.1. The space of functions C_α^m, $\alpha \in I\!R$, $m \in I\!N$ consists of all functions $f(x)$, $x > 0$, that can be represented in the form $f(x) = x^p f_1(x)$ with $p > \alpha$ and $f_1 \in C^m([0,\infty))$.

For the properties of the space C_α^m, see e.g. [13] or [28].

Theorem 3.1. *Let $n - 1 < \delta \le n$, $n \in I\!N$, $\alpha \ge -\beta(\gamma + 1)$ and $f \in C_\alpha^n$. Then the following relation between the E-K fractional derivative and E-K fractional integral of order δ holds true:*

$$(I_\beta^{\gamma,\delta}\, D_\beta^{\gamma,\delta}f)(x) \;=\; f(x) - \sum_{k=0}^{n-1} c_k x^{-\beta(1+\gamma+k)}, \tag{24}$$

$$c_k = \frac{\Gamma(n-k)}{\Gamma(\delta-k)}\lim_{x\to 0} x^{\beta(1+\gamma+k)}\prod_{i=k+1}^{n-1}(1+\gamma+i+\frac{1}{\beta}x\frac{d}{dx})(I_\beta^{\gamma+\delta,n-\delta}f)(x). \tag{25}$$

P r o o f. First we prove that $D_\beta^{\gamma,\delta}: C_\alpha^n \to C_\alpha$, under the conditions of the theorem. Indeed, using the same reasoning as in the proof of Theorem 2.2 and differentiating the integral representation of the function

$$f_2(x) = \int_0^1 \frac{(1-\tau)^{\delta-1}\tau^{\gamma+p/\beta}}{\Gamma(\delta)} f_1(x\tau^{1/\beta})d\tau,$$

n-times with respect to the variable x $(f_1 \in C^n([0,\infty))!)$ we can show that $I_\beta^{\gamma,\delta}: C_\alpha^n \to C_\alpha^n$ if $\alpha \geq -\beta(\gamma+1)-n$. It means that

$$I_\beta^{\gamma+\delta,n-\delta}: C_\alpha^n \to C_\alpha^n, \text{ if } \alpha \geq -\beta(\gamma+1+\delta)-n. \tag{26}$$

For a function f of the form $f(x) = x^p f_1(x)$ the relation

$$(a + bx\frac{d}{dx})(x^p f_1(x)) = x^p(af_1(x) + bpf_1'(x)) \tag{27}$$

is valid that implies the inclusion

$$\prod_{j=1}^n (\gamma + j + \frac{1}{\beta}x\frac{d}{dx})f \in C_\alpha \tag{28}$$

for $f(x) = x^p f_1(x),\ f_1 \in C^n[0,\infty)$. Combining the relations (26) and (28) with the definition of the Erélyi-Kober fractional derivative (19) we arrive at the relation $D_\beta^{\gamma,\delta}: C_\alpha^n \to C_\alpha,\ \alpha \geq -\beta(\gamma+1+\delta)-n$. Theorem 2.2 ensures now that the expression $(I_\beta^{\gamma,\delta} D_\beta^{\gamma,\delta} f)(x)$ is well defined on the space $C_\alpha^n,\ \alpha \geq -\beta(\gamma+1)$.

Let us now introduce an auxiliary function according to the rule

$$g(x) := (I_\beta^{\gamma,\delta} D_\beta^{\gamma,\delta} f)(x). \tag{29}$$

Using formula (21), we get then

$$(D_\beta^{\gamma,\delta} g)(x) = (D_\beta^{\gamma,\delta} I_\beta^{\gamma,\delta} D_\beta^{\gamma,\delta} f)(x) = (D_\beta^{\gamma,\delta} f)(x),$$

that implies the relation

$$(D_\beta^{\gamma,\delta}(f-g))(x) \equiv 0,\ x > 0. \tag{30}$$

The kernel of the operator $D_\beta^{\gamma,\delta}$ consists of all functions h that satisfy the relation

$$\prod_{k=0}^{n-1}(1+\gamma+k+\frac{1}{\beta}x\frac{d}{dx})y(x) \equiv 0,\ y(x) = (I_\beta^{\gamma+\delta,n-\delta}h)(x).$$

The linear homogeneous differential equation of the order n of the so-called hyper-Bessel-type (first introduced by Dimovski [4], see also [13, Ch.3])

$$\prod_{k=0}^{n-1}(1+\gamma+k+\frac{1}{\beta}x\frac{d}{dx})y(x)=0$$

possesses evidently the system of n linear independent solutions $y_k(x) = c_k x^{-\beta(1+\gamma+k)},\ k=0,...,n-1$, and the general solution of this equation can be written in the form

$$y(x)=\sum_{k=0}^{n-1} d_k x^{-\beta(1+\gamma+k)}. \tag{31}$$

Using the fact, that the Erdélyi-Kober fractional derivative is a left-inverse operator to the the Erdélyi-Kober fractional integral (see formula (21)) and the well known formula (see e.g. [13], [28])

$$(D_\beta^{\gamma,\delta} t^p)(x)=\frac{\Gamma(\gamma+\delta+p/\beta+1)}{\Gamma(\gamma+p/\beta+1)}x^p, \quad p+\beta(\gamma+1)>0,$$

we can solve the equation

$$(I_\beta^{\gamma+\delta,n-\delta}h)(x)=\sum_{k=0}^{n-1} d_k x^{-\beta(1+\gamma+k)}$$

by applying the operator $D_\beta^{\gamma+\delta,n-\delta}$ to both sides of this equation:

$$h(x)=\sum_{k=0}^{n-1} d_k \frac{\Gamma(n-k)}{\Gamma(\delta-k)}x^{-\beta(1+\gamma+k)}=\sum_{k=0}^{n-1} c_k x^{-\beta(1+\gamma+k)}.$$

This formula together with formula (30) leads to the representation

$$g(x)=f(x)-\sum_{k=0}^{n-1} c_k x^{-\beta(1+\gamma+k)}. \tag{32}$$

To finish the proof of the theorem, we need to determine the coefficients $c_k,\ k=0,1,...,n-1$ in the representation (32). To do this, we evaluate the expression $(I_\beta^{\gamma+\delta,n-\delta}g)(x)$, g being the auxiliary function (29), in two different ways. On the one hand, using the relation (7), we get

$$(I_\beta^{\gamma+\delta,n-\delta}g)(x)=(I_\beta^{\gamma+\delta,n-\delta}I_\beta^{\gamma,\delta}\,D_\beta^{\gamma,\delta}f)(x)=(I_\beta^{\gamma,n}\,D_\beta^{\gamma,\delta}f)(x) \tag{33}$$

$$=(I_\beta^{\gamma,n}\prod_{k=0}^{n-1}(1+\gamma+k+\frac{1}{\beta}t\frac{d}{dt})\,z)(x),\ z(t):=(I_\beta^{\gamma+\delta,n-\delta}f)(t). \tag{34}$$

Because $n \in I\!\!N$, the last expression can be evaluated by using integration by parts. For $n = 1$, we thus get:

$$(I_\beta^{\gamma,1}\,(1+\gamma+\frac{1}{\beta}t\frac{d}{dt})\,z)(x) \;=\; \beta x^{-\beta(\gamma+1)}\int_0^x t^{\beta(\gamma+1)-1}(1+\gamma+\frac{1}{\beta}t\frac{d}{dt})\,z(t)\,dt$$

$$= \beta x^{-\beta(\gamma+1)}\left(\frac{t^{\beta(\gamma+1)}}{\beta}\,z(t)\big|_0^x\right) = z(x) - x^{-\beta(\gamma+1)}\lim_{x\to 0} x^{\beta(\gamma+1)}z(x).$$

Then we can again apply the principle of the mathematical induction to prove the formula

$$(I_\beta^{\gamma,n}\prod_{k=0}^{n-1}(1+\gamma+k+\frac{1}{\beta}t\frac{d}{dt})\,z)(x) \;=\; z(x) - \sum_{k=0}^{n-1} a_k x^{-\beta(1+\gamma+k)}, \tag{35}$$

$$a_k = \lim_{x\to 0} x^{\beta(1+\gamma+k)}\prod_{i=k+1}^{n-1}(1+\gamma+i+\frac{1}{\beta}x\frac{d}{dx})z(x),$$

because the case $n = i+1$ can be reduced to the cases $n = i$ und $n = 1$ in the following way:

$$(I_\beta^{\gamma,i+1}\prod_{k=0}^{i}(1+\gamma+k+\frac{1}{\beta}t\frac{d}{dt})\,z)(x)$$

$$= (I_\beta^{\gamma+i,1}I_\beta^{\gamma,i-1}\prod_{k=0}^{i-1}(1+\gamma+k+\frac{1}{\beta}t\frac{d}{dt})(1+\gamma+i+\frac{1}{\beta}t\frac{d}{dt})\,z)(x)$$

$$= (I_\beta^{\gamma+i,1}I_\beta^{\gamma,i-1}\prod_{k=0}^{i-2}(1+\gamma+k+\frac{1}{\beta}t\frac{d}{dt})z_1)(x),\; z_1(t) := (1+\gamma+i+\frac{1}{\beta}t\frac{d}{dt})z(x).$$

On the other hand, we use representation (32) and the formula

$$(I_\beta^{\gamma,\delta}t^p)(x) = \frac{\Gamma(\gamma+p/\beta+1)}{\Gamma(\gamma+\delta+p/\beta+1)}x^p,\; p+\beta(\gamma+1) > 0,$$

to obtain

$$(I_\beta^{\gamma+\delta,n-\delta}g)(x) = z(x) - \sum_{k=0}^{n-1}\frac{\Gamma(\delta-k)}{\Gamma(n-k)}\,c_k x^{-\beta(1+\gamma+k)},\; z(x) = (I_\beta^{\gamma+\delta,n-\delta}f)(x).$$

Comparing now the last formula with the formulae (29), (32), (33), (34), (35) we arrive at the formulae (24) and (25) that completes the proof of the theorem. ∎

REMARK 3.1. The formulae (24) and (25) of Theorem 3.1 are similar to (but of course a little bit more complicated than) the corresponding formulae for the Riemann-Liouville fractional integral and derivative (see e.g. [25], [28]):

$$(I_{0+}^{\alpha} D_{0+}^{\alpha} f)(x) = \sum_{k=1}^{n} \frac{x^{\alpha-k}}{\Gamma(\alpha-k+1)} \lim_{x\to 0}(D_{0+}^{\alpha-k} f)(x),$$

where

$$(I_{0+}^{\alpha} f)(x) := \int_0^x \frac{(x-u)^{\alpha-1}}{\Gamma(\alpha)} f(u)du \tag{36}$$

is the Riemann-Liouville fractional integral of order $\alpha,\ n-1<\alpha\le n,\ n\in I\!N$ and

$$(D_{0+}^{\alpha} f)(x) := \left(\frac{d}{dx}\right)^n (I_{0+}^{n-\alpha} f)(x) \tag{37}$$

is the Riemann-Liouville fractional derivative of order $\alpha,\ n-1<\alpha\le n,\ n\in I\!N$. The constants $\lim\limits_{x\to 0}(D_{0+}^{\alpha-k} f)(x),\ k=1,...,n$ appear also in the formula for the Laplace transform of the Riemann-Liouville fractional derivative of order $\alpha,\ n-1<\alpha\le n,\ n\in I\!N$ and, as a consequence, as the initial conditions in the initial-value problems for the fractional differential equations with the Riemann-Liouville fractional derivatives. Because there is no known physical interpretation for the expressions $\lim\limits_{x\to 0}(D_{0+}^{\alpha-k} f)(x),\ k=1,...,n$, the Riemann-Liouville fractional derivatives are not so frequently used for the modeling of the applied problems. In particular, this was one of the reasons for introduction of the Caputo fractional derivative in the form

$$({}_*D_{0+}^{\alpha} f)(x) := (I_{0+}^{n-\alpha} f^{(n)})(x). \tag{38}$$

The Caputo derivative applied to a constant function is equal to zero and the Laplace transform of the Caputo derivative of order $\alpha,\ n-1<\alpha\le n,\ n\in I\!N$ of a function f is expressed in terms of the Laplace transform of the function f and the initial values $f(0),...,f^{(n-1)}(0)$, that can be interpreted in applications in a suitable way.

As we have seen in Theorem 3.1, the constants

$$\lim_{x\to 0} x^{\beta(1+\gamma+k)} \prod_{i=k+1}^{n-1} (1+\gamma+i+\frac{1}{\beta}x\frac{d}{dx})(I_\beta^{\gamma+\delta,n-\delta} f)(x)$$

that appear in the formula for the composition of the E-K fractional integral and E-K fractional derivative $I_\beta^{\gamma,\delta}\, D_\beta^{\gamma,\delta}$ (and will appear in the formula for the Laplace transform of the E-K fractional derivative, too) cannot be well interpreted in the applications. Our motivation behind the introduction and investigation of the Caputo-type modification of the Erdélyi-Kober fractional derivative is in fact the same as the one by Caputo: we are trying to define an operator that is similar to the classical Erdélyi-Kober fractional derivative, but allows the more or less traditional form of initial conditions while considering fractional differential equations with such type of fractional derivative.

4. Caputo-type modification of the Erdélyi-Kober derivatives

In this section we define and give some basic properties of the Caputo-type modification of the Erdélyi-Kober fractional derivatives. Let $n-1 < \delta \le n,\ n \in I\!N$ and $m-1 < \alpha \le m,\ m \in I\!N$. The operator

$$({}_*D_\beta^{\gamma,\delta} f)(x) := (I_\beta^{\gamma+\delta,n-\delta} \prod_{k=0}^{n-1}(1+\gamma+k+\frac{1}{\beta}t\frac{d}{dt})f)(x) \tag{39}$$

is called the right-hand sided Caputo-type modification of the Erdélyi-Kober fractional derivative of order δ. The left-hand sided Caputo-type modification of the E-K fractional derivative of order α is analogously defined by

$$({}_*P_\beta^{\tau,\alpha} f)(x) := (J_\beta^{\tau+\alpha,m-\alpha} \prod_{j=0}^{m-1}(\tau+j-\frac{1}{\beta}t\frac{d}{dt})f)(x). \tag{40}$$

In the formulae (39) and (40) the operators $I_\beta^{\gamma,\delta}$ and $J_\beta^{\tau,\alpha}$ are the right- and the left-hand sided E-K fractional integrals of orders δ and α, respectively, defined by (4) and (5). In the rest of the section, we deal with the right-hand sided Caputo-type modification of the Erdélyi-Kober derivative ${}_*D_\beta^{\gamma,\delta}$. The case of the left-hand sided derivative can be considered by analogy. The main result of the section is given by the following

Theorem 4.1. *Let $n-1 < \delta \le n,\ n \in I\!N,\ \alpha \ge -\beta(\gamma+\delta+1)$ and $f \in C_\alpha^n$. Then the following relation between the Caputo-type modification of the Erdélyi-Kober fractional derivative and the Erdélyi-Kober fractional integral of order δ holds true:*

$$(I_\beta^{\gamma,\delta}\, {}_*D_\beta^{\gamma,\delta} f)(x) = f(x) - \sum_{k=0}^{n-1} p_k x^{-\beta(1+\gamma+k)}, \tag{41}$$

$$p_k = \lim_{x\to 0} x^{\beta(1+\gamma+k)} \prod_{i=k+1}^{n-1}(1+\gamma+i+\frac{1}{\beta}x\frac{d}{dx})f(x). \tag{42}$$

P r o o f. First we show that under the conditions of the theorem the function ${}_*D_\beta^{\gamma,\delta} f$ belongs to the space C_α if $f \in C_\alpha^n$. To prove this, we first refer to the relation (27), that implies that

$$\prod_{k=0}^{n-1}(1+\gamma+j+\frac{1}{\beta}x\frac{d}{dx})f \in C_\alpha \quad \text{if} \quad f \in C_\alpha^n.$$

The inclusion ${}_*D_\beta^{\gamma,\delta} f \in C_\alpha,\ \alpha \geq -\beta(\gamma+\delta+1)$ follows then directly from Theorem 2.2. Using the definition of the Caputo-type modification of the E-K fractional derivative and the property (7) of the E-K fractional integral, we obtain the relation

$$(I_\beta^{\gamma,\delta}\, {}_*D_\beta^{\gamma,\delta}\, f)(x) = (I_\beta^{\gamma,\delta} I_\beta^{\gamma+\delta,n-\delta} \prod_{k=0}^{n-1}(1+\gamma+j+\frac{1}{\beta}t\frac{d}{dt})f)(x) \tag{43}$$

$$= (I_\beta^{\gamma,n} \prod_{k=0}^{n-1}(1+\gamma+j+\frac{1}{\beta}t\frac{d}{dt})f)(x).$$

The last expression can be represented according to the formula (35) in the form

$$(I_\beta^{\gamma,n} \prod_{k=0}^{n-1}(1+\gamma+k+\frac{1}{\beta}t\frac{d}{dt})\, f)(x) \;=\; f(x) - \sum_{k=0}^{n-1} p_k x^{-\beta(1+\gamma+k)},$$

$$p_k = \lim_{x\to 0} x^{\beta(1+\gamma+k)} \prod_{i=k+1}^{n-1}(1+\gamma+i+\frac{1}{\beta}x\frac{d}{dx})f(x),$$

that completes the proof of the theorem. ■

Remark 4.1. As expected, the constants

$$\lim_{x\to 0} x^{\beta(1+\gamma+k)} \prod_{i=k+1}^{n-1}(1+\gamma+i+\frac{1}{\beta}x\frac{d}{dx})f(x)$$

in the formulae (41) and (42) depend only on the ordinary derivatives of the function f with some power weights and do not depend on the limit values of the fractional integrals at the point $x=0$.

Now let us consider the question about the conditions under which the Caputo-type modification of the Erdélyi-Kober fractional derivative ${}_*D_\beta^{\gamma,\delta}$ coincides with the Erdélyi-Kober fractional derivative $D_\beta^{\gamma,\delta}$ on the functional space $C_\alpha^n,\ n-1<\delta\leq n,\ n\in I\!N$.

Theorem 4.2. *The Caputo-type modification of the E-K fractional derivative ${}_*D_\beta^{\gamma,\delta}$ coincides with the E-K fractional derivative $D_\beta^{\gamma,\delta}$ for a function $f \in C_\alpha^n,\ n-1 < \delta \le n,\ n \in I\!N,\ \alpha \ge -\beta(\gamma+1)$, if and only if the conditions*

$$\lim_{x\to 0} x^{\beta(1+\gamma+k)} \prod_{i=k+1}^{n-1} (1+\gamma+i+\frac{1}{\beta}x\frac{d}{dx})f(x)$$
$$= \frac{\Gamma(n-k)}{\Gamma(\delta-k)} \lim_{x\to 0} x^{\beta(1+\gamma+k)} \prod_{i=k+1}^{n-1} (1+\gamma+i+\frac{1}{\beta}x\frac{d}{dx})(I_\beta^{\gamma+\delta,n-\delta}f)(x)$$

are fulfilled for all $k = 0, 1, ..., n-1$.

P r o o f. For a function $f \in C_\alpha^n,\ n-1 < \delta \le n,\ n \in I\!N,\ \alpha \ge -\beta(\gamma+1)$ both the Caputo-type modification of the Erdélyi-Kober fractional derivative ${}_*D_\beta^{\gamma,\delta}$ and the Erdélyi-Kober fractional derivative $D_\beta^{\gamma,\delta}$ exist as we showed in the proofs of Theorems 4.1 and 3.1, respectively. Now let

$$({}_*D_\beta^{\gamma,\delta}f)(x) \equiv (D_\beta^{\gamma,\delta}f)(x),\ x > 0.$$

Applying the E-K operator $I_\beta^{\gamma,\delta}$ to both sides of the above relation, we get

$$(I_\beta^{\gamma,\delta}\,{}_*D_\beta^{\gamma,\delta}f)(x) \equiv (I_\beta^{\gamma,\delta}D_\beta^{\gamma,\delta}f)(x),\ \ x > 0.$$

According to Theorem 4.1,

$$(I_\beta^{\gamma,\delta}\,{}_*D_\beta^{\gamma,\delta}f)(x) \ = \ f(x) - \sum_{k=0}^{n-1} p_k x^{-\beta(1+\gamma+k)},$$
$$p_k = \lim_{x\to 0} x^{\beta(1+\gamma+k)} \prod_{i=k+1}^{n-1} (1+\gamma+i+\frac{1}{\beta}x\frac{d}{dx})f(x). \tag{44}$$

Theorem 3.1 states that

$$(I_\beta^{\gamma,\delta}\,D_\beta^{\gamma,\delta}f)(x) \ = \ f(x) - \sum_{k=0}^{n-1} c_k x^{-\beta(1+\gamma+k)},$$
$$c_k = \frac{\Gamma(n-k)}{\Gamma(\delta-k)} \lim_{x\to 0} x^{\beta(1+\gamma+k)} \prod_{i=k+1}^{n-1} (1+\gamma+i+\frac{1}{\beta}x\frac{d}{dx})(I_\beta^{\gamma+\delta,n-\delta}f)(x). \tag{45}$$

From the last three relations it follows that

$$\sum_{k=0}^{n-1} p_k x^{-\beta(1+\gamma+k)} \ = \sum_{k=0}^{n-1} c_k x^{-\beta(1+\gamma+k)},$$

and, because the functions $\{x^{-\beta(1+\gamma+k)},\ k=0,...,n-1\}$ are linear independent, we have

$$p_k = c_k,\ k = 0, 1, ..., n-1,$$

where the coefficients p_k and c_k are defined by (44) and (45), respectively, what we wanted to prove.

Now, let the conditions of the theorem be fulfilled, i.e. $p_k = c_k,\ k = 0, 1, ..., n-1$, where the coefficients p_k and c_k are defined by (44) and (45), respectively.

Then it follows from Theorems 3.1 and 4.1 that

$$(I_\beta^{\gamma,\delta}\, {}_*D_\beta^{\gamma,\delta} f)(x) \equiv (I_\beta^{\gamma,\delta} D_\beta^{\gamma,\delta} f)(x),\ x > 0. \tag{46}$$

The identity

$$({}_*D_\beta^{\gamma,\delta} f)(x) \equiv (D_\beta^{\gamma,\delta} f)(x),\ x > 0$$

follows then from the formula (21) by applying the Erdélyi-Kober fractional derivative $D_\beta^{\gamma,\delta}$ to both sides of the relation (46).

■

The last problem we consider in this section is if the Caputo-type modification of the E-K fractional derivative is a left-inverse operator to the E-K fractional integral (4), as it was the case for the E-K fractional derivative (see formula (21)). The corresponding result is given in the following important

THEOREM 4.3. *The Caputo-type modification* ${}_*D_\beta^{\gamma,\delta}$ *of the Erdélyi-Kober fractional derivative is a left-inverse operator to the Erdélyi-Kober fractional integral for the functions from the functional space* $C_\alpha,\ \alpha \geq -\beta(\gamma+1)$, *i.e.,*

$$({}_*D_\beta^{\gamma,\delta}\, I_\beta^{\gamma,\delta} f)(x) \equiv f(x),\ f \in C_\alpha. \tag{47}$$

P r o o f. The direct proof of the relation (47) is very long and complicated, so we prefer to employ an indirect approach. Namely, let us introduce an auxiliary function g according to the rule

$$g(x) := (I_\beta^{\gamma,\delta} f)(x). \tag{48}$$

We shall now show that for the function g all conditions of Theorem 4.2 are fulfilled, so that

$$({}_*D_\beta^{\gamma,\delta} g)(x) \equiv (D_\beta^{\gamma,\delta} g)(x).$$

This last formula together with the formula (21) leads to

$$({}_*D_\beta^{\gamma,\delta} I_\beta^{\gamma,\delta} f)(x) \equiv ({}_*D_\beta^{\gamma,\delta} g)(x) \equiv (D_\beta^{\gamma,\delta} g)(x) \equiv (D_\beta^{\gamma,\delta} I_\beta^{\gamma,\delta} f)(x) \equiv f(x).$$

Let us prove now that

$$p_k = c_k = 0, \ k = 0, 1, ..., n-1,$$

where the coefficients p_k and c_k are defined by (44) and (45), respectively, and the function f is replaced by the function $g = (I_\beta^{\gamma,\delta} f)(x)$. We begin with the coefficients p_k. For $k = n-l, \ l = 1, ..., n$, we get

$$p_{n-l} = \lim_{x\to 0} x^{\beta(\gamma+1+n-l)} \prod_{i=n-l+1}^{n-1} (1+\gamma+i+\frac{1}{\beta}x\frac{d}{dx})g(x)$$

$$= \lim_{x\to 0} x^{\beta(\gamma+1+n-l)} x^p g_l(x) = 0,$$

because $\beta(\gamma+1+n-l)+p > \beta(\gamma+n)+\alpha \geq \beta(\gamma+n)-\beta(\gamma+1) = \beta(n-l) \geq 0$ and the functions $g_l, \ l = 1, ..., n$ are from the space $C([0,\infty))$, due to Theorem 2.2 and relation (27).

To calculate the coefficients $c_k, \ k = 0, 1, ..., n-1$ we employ the formula (7) for the composition of two Erdélyi-Kober fractional integrals:

$$(I_\beta^{\gamma+\delta,n-\delta} g)(x) = (I_\beta^{\gamma+\delta,n-\delta} I_\beta^{\gamma,\delta} f)(x) = (I_\beta^{\gamma,n} f)(x).$$

Then

$$c_k = \frac{\Gamma(n-k)}{\Gamma(\delta-k)} \lim_{x\to 0} x^{\beta(1+\gamma+k)} \prod_{i=k+1}^{n-1} (1+\gamma+i+\frac{1}{\beta}x\frac{d}{dx})(I_\beta^{\gamma,n} f)(x)$$

$$= \frac{\Gamma(n-k)}{\Gamma(\delta-k)} \lim_{x\to 0} x^{\beta(\gamma+1+k)} x^p f_k(x) = 0,$$

because $\beta(\gamma+1+k)+p > \beta(\gamma+1+k)+\alpha \geq \beta(\gamma+1+k)-\beta(\gamma+1) = \beta k \geq 0$ and the functions $f_k, \ k = 0, ..., n-1$ are from the space $C([0,\infty))$, due to Theorem 2.2 and relation (27). The relation

$$p_k = c_k = 0, \ k = 0, 1, ..., n-1,$$

is now proved, so that we can use the reasoning already presented at the beginning, to complete the proof of the theorem. ∎

REMARK 4.2. In the paper, we give the definitions and some of the basic properties of the Caputo-type modification of the Erdélyi-Kober fractional derivative. As has been mentioned in the Introduction, this type of operators already found its first applications. We hope, that the Caputo-type modification of the Erdélyi-Kober fractional derivative can be successfully applied in other areas of research and modelling mainly due to the formulae (41) and (42) that depend only on the ordinary derivatives of the function f with some power weights and do not depend on the limit values of the fractional integrals at the point $x = 0$ as it was the case for the classical Erdélyi-Kober fractional derivative. Compared with the Riemann-Liouville fractional derivative a potential advantage of the Caputo-type modification of the Erdélyi-Kober fractional derivative for modelling of the applied problems lie in the fact that in addition to the derivatives order δ it has two additional parameters (γ and δ) that can be interpreted as two additional freedom degrees in the Fractional Models. Among other directions of research connected with the new definition of a fractional derivative we mention the applications of this operators to integral equations, fractional differential equations with the Caputo-type modification of the Erdélyi-Kober fractional derivative, operational calculus for this operator, Leibniz-type rules, integral transforms of the new fractional derivative and special functions related to it. These open problems will be considered elsewhere.

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References

[1] E. B u c k w a r, Yu. F. L u c h k o, Invariance of a partial differential equation of fractional order under the Lie group of scaling transformations. *J. Math. Anal. Appl.*, **227** (1998), 81-97.

[2] M. C a p u t o, The Green function of the diffusion of fluids in porous media with memory. *Rend. Fis. Acc. Lincei* (Ser. 9) **7** (1996), 243-250.

[3] A.V. C h e c h k i n, R. G o r e n f l o, I.M. S o k o l o v, Fractional diffusion in inhomogeneous media. *J. Phys. A, Math. Gen.* **38** (2005), 679-684.

[4] I.H. D i m o v s k i, Operational calculus for a class of differential operators. *Compt. rend. Acad. bulg. Sci.* **19** (1966), 1111-1114.

[5] A. F r e e d, K. D i e t h e l m, Yu. L u c h k o, *Fractional-order viscoelasticity (FOV): Constitutive development using the fractional calculus*, NASA report TM 2002-211914, NASA's Glenn Research Center: Brook Rark, Ohio, (2002).

[6] M. G i o n a, H.E. R o m a n, A theory of transport phenomena in disordered systems. *The Chemical Eng. Journal* **49** (1992), 1-10.

[7] R. G o r e n f l o, Yu. F. L u c h k o, F. M a i n a r d i, Wright functions as scale-invariant solutions of the diffusion-wave equation. *J. Computational and Applied Mathematics* **11** (2000), 175-191.

[8] R. H i l f e r, Exact solutions for a class of fractal time random walks, *Fractals* **3** (1995), 211-216.

[9] R. H i l f e r (ed.), *Applications of Fractional Calculus in Physics.* World Scientific, Singapore, (2000).

[10] R. H i l f e r, On fractional diffusion and continuous time random walks. *Physica A* **329** (2003), 35-40.

[11] S.L. K a l l a, V.S. K i r y a k o v a, An H-function generalized fractional calculus based upon compositions of Erdélyi-Kober operators in L_p. *Math. Japonica* **35** (1990), 1-21.

[12] A.A. K i l b a s, H.M. S r i v a s t a v a, J.J. T r u j i l l o, *Theory and Applications of Fractional Differential Equation.* Elsevier, Amsterdam (2006).

[13] V. K i r y a k o v a, *Generalized Fractional Calculus and Applications.* Pitman Research Notes in Math. **301**, Longman, Harlow - J. Wiley, N. York (1994).

[14] H. K o b e r, On fractional integrals and derivatives. *Quart. J. Math. Oxford*, Ser. ll (1940), 193 -211.

[15] Yu. F. L u c h k o, R. G o r e n f l o, Scale-invariant solutions of a partial differential equation of fractional order. *Fract. Calc. Appl. Anal.* **1**, No 1 (1998), 63-78.

[16] F. M a i n a r d i, Fractional diffusive waves in viscoelastic solids. In: J.L. Wagner and F.R. Norwood (Eds.), *IUTAM Symposium - Nonlinear Waves in Solids* (ASME/AMR, Fairfield NJ, 1995), 93-97.

[17] F. M a i n a r d i, M. T o m i r o t t i, Seismic pulse propagation with constant Q and stable probability distributions. *Annali di Geofisica* **40** (1997), 1311-1328.

[18] F. M a i n a r d i, R. G o r e n f l o, E. S c a l a s, A fractional generalization of the Poisson processes. *Vietnam J. Math.* **32** (2004), 53-64.

[19] F. M a i n a r d i, Yu. L u c h k o, G. P a g n i n i, The fundamental solution of the space-time fractional diffusion equation. *Fract. Calc. Appl. Anal.* **4**, No 2 (2001), 153-192.

[20] D. M a t i g n o n, G. M o n t s e n y (Eds.), *Fractional Differential Systems: Models, Methods and Applications*, Proc. Colloquium FDS'98. ESAIM (European Ser. Appl. & Ind. Maths) Proceedings **5** (1996).

[21] R. M e t z l e r, W. G. G l ö c k l e, T. F. N o n n e n m a c h - e r, Fractional model equation for anomalous diffusion. *Physica A* **211** (1994), 13-24.

[22] R.R. N i g m a t u l l i n, The realization of the generalized transfer equation in a medium with fractal geometry. *Phys. Stat. Sol. B* **133** (1986), 425-430.

[23] A.C. P i p k i n, *Lectures on Viscoelastic Theory*. Springer Verlag, New York (1986).

[24] I. P o d l u b n y, *Fractional Differential Equations.* Academic Press, San Diego (1999).

[25] S.G. S a m k o, A.A. K i l b a s, O.I. M a r i c h e v, *Fractional Integrals and Derivatives: Theory and Applications.* Gordon and Breach, New York (1993).

[26] I.N. S n e d d o n, The use in mathematical analysis of Erdelyi-Kober operators and some of their applications. In: *Fractional Calculus and Its Applications*, Proc. Internat. Conf. Held in New Haven, = *Lecture Notes in Math.* **457**, Springer, N. York (1975), 37-79.

[27] E.M. W r i g h t, The asymptotic expansion of the generalized hypergeometric function. *Journal London Math. Soc.* **10** (1935), 287-293.

[28] S. B. Y a k u b o v i c h, Yu. F. L u c h k o, *The Hypergeometric Approach to Integral Transforms and Convolutions.* Mathematics and its Appl. **287**, Kluwer Acad. Publ., Dordrecht-Boston-London (1994).

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